L07 Jan 20 Continuity

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Continuity of $f:(X,\mathcal{T}_X)\longrightarrow(Y,\mathcal{T}_Y)$

* At a point $x_0\in X$
  $\forall\ V\in\mathcal{T}_Y$ with $f(x_0)\in V$
  $\exists\ U\in\mathcal{T}_X$ with $x_0\in U$, $f(U)\subset V$
* Everywhere
  $\forall\ V\in\mathcal{T}_Y$, $f^{-1}(V)\in\mathcal{T}_X$

From the definition f is always continuous

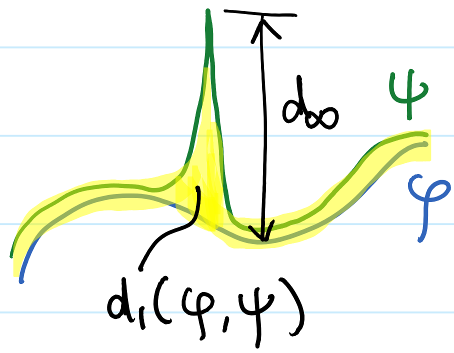
* if $\mathcal{T}_X$ is discrete, or
* if $\mathcal{T}_Y$ is indiscrete

Example. $X=\{\text{continuous functions on } [a,b]\}$

$$d_1(\varphi,\psi)=\int_a^b|\varphi(t)-\psi(t)|\,dt$$

$$d_\infty(\varphi,\psi)=\sup\{|\varphi(t)-\psi(t)|: t\in[a,b]\}$$

d_∞ small $\Longrightarrow$ d_1 small
$\not\Longleftarrow$



Thus,

$id:(X,d_\infty)\longrightarrow(X,d_1)$ is continuous

$id:(X,d_1)\longrightarrow(X,d_\infty)$ not continuous

Theorem. For $f: X \longrightarrow Y$, these are equivalent

1. f is continuous at each $x \in X$
2. $\forall V \in \mathcal{T}_Y \quad f^{-1}(V) \in \mathcal{T}_X$
3. $\forall V \in \mathcal{B}_Y \quad f^{-1}(V) \in \mathcal{T}_X$
4. $\forall A \subset X \quad f(\bar{A}) \subset \overline{f(A)}$
5. $\forall B \subset Y \quad \overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$
6. $\forall$ closed $H \subset Y \quad f^{-1}(H)$ is closed in X

1 $\Leftrightarrow$ 2 and 2 $\Rightarrow$ 3 trivial

3 $\Rightarrow$ 2, 6 $\Rightarrow$ 1 obvious

4 $\Rightarrow$ 5, 5 $\Rightarrow$ 6 elementary

$A = f^{-1}(B)$ $\qquad B = H = \bar{B}$

Only remaining is 1 or 2 or 3 $\Rightarrow$ 4

Take any $x \in \bar{A}$, to show $f(x) \in \overline{f(A)}$

Need $\forall V \in \mathcal{T}_Y$ with $f(x) \in V$, $V \cap f(A) \neq \emptyset$

By continuity of f at x, $\exists U \in \mathcal{T}_X$, $x \in U$

and $f(U) \subset V$

Since $x \in \bar{A}$, for this $U \in \mathcal{T}_X$ with $x \in U$,

$$U \cap A \neq \emptyset, \quad \therefore \exists a \in U \cap A$$

$$f(a) \in f(U) \subset V \quad \therefore V \cap f(A) \neq \emptyset \qquad \square$$

Recall Let $(X,\mathcal{T})$ be a topological space and $A \subset X$. Then $\mathcal{T}|_A = \{ G \cap A : G \in \mathcal{T} \}$ is called the subspace topology on A.

Proposition Let $X = \bigcup_{\alpha \in I} G_\alpha$ where $G_\alpha \in \mathcal{T}$.
If $f_\alpha : G_\alpha \longrightarrow Y$ are continuous under the subspace topologies such that

$$f_\alpha = f_\beta \text{ on } G_\alpha \cap G_\beta$$

then $f : X \longrightarrow Y$ is well-defined and continuous

$$f(x) = f_\alpha(x) \text{ if } x \in G_\alpha$$

Proposition Let $X = A \cup B$ where A, B are closed.
If $f|_A$ and $f|_B$ are continuous under the subspace topologies then f is continuous.
Let $H \subset Y$ be closed and consider $f^{-1}(H)$.

$$f^{-1}(H) = \left(f^{-1}(H) \cap A\right) \cup \left(f^{-1}(H) \cap B\right)$$

$$= (f|_A)^{-1}(H) \cup (f|_B)^{-1}(H)$$

Therefore $f^{-1}(H)$ is closed.

Qu. What is the key difference between open sets and closed sets?

Qu. Give a bad example for closed sets.

Uniquely determined on dense set

Let $A \subset X$ be dense and Y is Hausdorff
$f, g: X \longrightarrow Y$ are continuous.
If $f|_A = g|_A$ then $f \equiv g$ on X.

Take any $x \in X = \bar{A}$ and any nbhds V_1, V_2 of $f(x), g(x)$ respectively.
By continuity of f and g,
$f^{-1}(V_1)$, $g^{-1}(V_2)$ are nbhds of x

and so is $U = f^{-1}(V_1) \cap g^{-1}(V_2)$
Since $\bar{A} = X$, $A \cap U \neq \emptyset$, $\exists\ a \in A \cap U$
Thus $f(a) = g(a) \in V_1 \cap V_2$
We've shown any nbhds of $f(x)$ and $g(x)$ must intersect each other. In a Hausdorff space, this occurs only if $f(x) = g(x)$. $\square$

Definition $f: (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is

* a homeomorphism if f^{-1} exists and is also continuous
* an open mapping if $\forall\ U \in \mathcal{T}_X$, $f(U) \in \mathcal{T}_Y$

Recall x is a cluster point of A, i.e., $x \in A'$

if $\forall \underbrace{U \in \mathcal{T} \text{ with } x \in U}_{\Updownarrow \; U \in \mathcal{U}_x \text{, local base at } x}$, $\underbrace{U \cap A \setminus \{x\} \neq \emptyset}_{\exists\, a \in A \cap U,\, a \neq x}$

Local base may not have a linear order

such as $U_1 < U_2 < U_3 < \cdots < U_n < \cdots$

$ \; \overset{\cap}{a_1} \quad \overset{\cap}{a_2} \quad \overset{\cap}{a_3} \quad \cdots \quad \overset{\cap}{a_n} \longrightarrow x$

e.g. X = metric space, $U_n = B(x, \frac{1}{n})$, "$<$" $\Leftrightarrow$ $\supset$.

Cluster is essentially sequence convergence.

Convergence A sequence in $(X, \mathcal{T})$ is a

mapping $\mathbb{N} \longrightarrow X : n \longmapsto x_n$,

denoted by $(x_n)_{n=1}^{\infty}$ or $\{x_n\}_{n=1}^{\infty}$

It converges to $x \in X$ if

$\forall$ nbhd U of x, ... $x_n \in U$

fill in the dots

Also says x is a limit of $(x_n)_{n=1}^{\infty}$

Proposition Limit of a sequence is unique if the space $(X, \mathcal{T})$ is Hausdorff.

As mentioned, expect related to cluster

Proposition Let $A \subset X$ and $a_n \in A$.

If $a_n \to x$ then $x \in \overline{A}$

Idea. Take any nbhd U of x,

$$\vdots$$

$$\exists\, a_N \in U, \quad \therefore \quad U \cap A \neq \emptyset$$

Remark If a_n is a distinct sequence then $x \in A'$

The converse is not true.

Proposition Let (X,d) be a metric space.

$x \in \overline{A} \Rightarrow \exists\, a_n \in A$ such that $a_n \to x$.

Take the local base $B(x, \frac{1}{n})$ and

$$a_n \in B(x, \tfrac{1}{n}) \cap A$$

Then obviously, $a_n \to x$

Qu. What if X is only C_I?

Let $\mathcal{U}_x = \{U_n : n \in \mathbb{N}\}$ be a countable local base. Consider $V_n = U_1 \cap \cdots \cap U_n$.

Qu. Can $(a_n)_{n=1}^{\infty}$ be distinct if $x \in A'$?

In Calculus, we thought of continuity by

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$$

This is partially true in topology

Proposition. Let $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$.

If f is continuous at $x_0 \in X$ then

$\forall\ x_n \longrightarrow x_0$ in X, $f(x_n) \longrightarrow f(x_0)$ in Y

Idea. Take any $V \in \mathcal{T}_Y$ with $f(x_0) \in V$

By continuity of f, $\exists\ U \in \mathcal{T}_X$, $x_0 \in U \subset f^{-1}(V)$

$\exists\ N \in \mathbb{N}$ such that $\forall\ n \geq N$ $x_n \in U \subset f^{-1}(V)$

and so $f(x_n) \in V$ $\square$

Again, for the converse, metric is required

Assume f is not continuous at x_0

Construct $x_n \in B(x_0, \frac{1}{n})$ such that

⋮

So, $\exists\ x_n \longrightarrow x_0$ in X but $f(x_n) \not\longrightarrow f(x_0)$

Qu. Think about whether G_I for X is enough.